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# Sharp Orders of Convergence in the Random Central Limit Theorem

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## 1. INTRODUCTION AND NOTATION

Let $(\Omega, \mathscr{A}, P)$ be a probability space. Let $\mathbb{R}^k$ be endowed with the euclidean norm $|\ |$ and denote by $\mathscr{L}_p(\Omega, \mathscr{A}, P, \mathbb{R}^k)$ the system of all random vectors $X\colon \Omega \to \mathbb{R}^k$ with $E(|X|^p) < \infty$.

Let $X_n \in \mathscr{L}_2(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, be a sequence of i.i.d. random vectors with positive definite covariance matrix $V$. Put $S_n = V^{-1/2} \sum_{\nu=1}^n (X_\nu - E(X_\nu))$ and denote by $\Phi_{O,I}$ the standard normal distribution or its distribution function in $\mathbb{R}^k$. Let $\tau_n\colon \Omega \to \mathbb{N}$, $n \in \mathbb{N}$, and $\tau\colon \Omega \to (0, \infty)$ be $\mathscr{A}$-measurable.

The classical random central limit theorem states that

$$H_n(\tau) := \sup_{t \in \mathbb{R}^k} \left| P\left\{ \frac{S_{\tau_n}}{\tau_n^{1/2}} \leqslant t \right\} - \Phi_{O,I}(t) \right| = o(1)$$

if $\tau_n/n \to \tau$ in probability, or equivalently, if

$$(1) \qquad P\left\{ \left| \frac{\tau_n}{n\tau} - 1 \right| > \varepsilon \right\} = o(1) \qquad \text{for all} \quad \varepsilon > 0.$$

This was proven first for a constant and for a discrete limit function $\tau$ by Renyi [21]. For an arbitrary limit function $\tau$ it was proven by Blum, Hanson and Rosenblatt [4]. The important role of the random central



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limit theorem for various fields of applications such as sequential analysis, Monte Carlo methods, and the theory of Random walks and Markov chains is nowadays well known.

Hence it seems desirable and worthwhile to find convergence rates for $H_n(\tau)$. Several papers have been devoted to this (see [13, 14] and the literature cited there). Hitherto, rates of convergence for $H_n(\tau)$ were known only for constant limit functions $\tau$, or a little bit more general, for limit functions $\tau$ which are independent of the whole process.

For a constant $\tau$ (and $X_n \in \mathscr{L}_3$) it was proven, e.g., in [13, 14], that the sharpened "type (1)"-version

$$P\left\{\left|\frac{\tau_n}{n\tau} - 1\right| > \varepsilon_n\right\} = O(\varepsilon_n^{1/2}) \tag{2}$$

implies $H_n(\tau) = O(\varepsilon_n^{1/2})$, where $1/n \leqslant \varepsilon_n \downarrow 0$ (a result which was applied, e.g., in [10, 12] and extended to other processes in [1, 2, 11, 23]). An example given in [14] shows that this result fails for a non-constant limit function $\tau$: the convergence order of $H_n(\tau)$ can be made arbitrarily slow, even with a two valued limit function $\tau: \Omega \to \{1, 2\}$ and with $\tau_n = n\tau$ (whence (2) holds for each sequence $\varepsilon_n$). It is the purpose of this paper to close this wide gap between constant and non-constant limit functions $\tau$. Furthermore, we consider instead of $H_n(\tau)$ the larger

$$\hat{H}_n(\tau) := \sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_{\tau_n}}{\tau_n^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right|,$$

where $\mathscr{C}$ is the system of all convex Borel-measurable sets of $\mathbb{R}^k$. Some of our auxiliary lemmas in Section 4 (Lemmas 4.1–4.4) are needed only to handle the class of convex sets and can be omitted if one is only interested in distribution functions (i.e., in $H_n(\tau)$ instead of $\hat{H}_n(\tau)$).

To deal with non-constant $\tau$, the problem is to find a reasonable condition for $\tau$, which guarantees—together with (2)—a good convergence order for $\hat{H}_n(\tau)$.

It turns out that the "one-sided" Hausdorff-metric between $\sigma$-fields $\sigma(\tau)$ and $\sigma(X_1, ..., X_n)$ allows one to formulate such a condition (where $\sigma(Y)$ is the $\sigma$-field generated by $Y$).

If $\mathscr{A}_0, \mathscr{B}_0 \subset \mathscr{A}$ are $\sigma$-fields, define

$$d(A, \mathscr{B}_0) = \inf_{B \in \mathscr{B}_0} P(A \,\Delta B), \qquad \rho(\mathscr{A}_0, \mathscr{B}_0) = \sup_{A \in \mathscr{A}_0} d(A, \mathscr{B}_0).$$

Observe that $\rho(\mathscr{A}_0, \mathscr{B}_0) + \rho(\mathscr{B}_0, \mathscr{A}_0)$ is the Hausdorff-metric between $\mathscr{A}_0$, $\mathscr{B}_0$, if the sub-$\sigma$-fields are completed; otherwise we have only a pseudometric in general.

The distances $d(A, \sigma(X_1, ..., X_n))$ have been used in [15–17] to obtain convergence orders and asymptotic expansions for the conditional central limit theorem of Renyi. The Hausdorff distance between $\sigma$-fields or $\sigma$-lattices was studied by Boylan [5], Neveu [20], Rogge [22], Brunk [6], and Mukerjee [19] and used to obtain uniform convergence rates in martingale theorems.

In this paper we use the Hausdorff distances to get the following: Let $0<\alpha<\frac{1}{2}$, $\beta \in \mathbb{R}$, and $\inf \tau(\Omega)>0$. Then condition (2) and

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = O(n^{-\alpha}(\lg n)^{\beta}) \tag{3}$$

imply that

$$\hat{H}_n(\tau) = O(\varepsilon_n^{1/2}) + O(n^{-\alpha}(\lg n)^{\alpha+\beta}) \tag{4}$$

(see Theorem 2.1). An essential tool for the proof of this result is an inequality for the Hausdorff-metric of $\sigma$-fields, proven in [18].

Let us remark that condition (3) is for instance fulfilled for each stopping time $\tau$ with $E(\sqrt{\tau})<\infty$, since in this case $\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) \leqslant \sup_B P\{(\tau \in B) \triangle (\tau \in B \cap \{1, ..., n\})\} \leqslant P(\tau > n\} \leqslant (1/\sqrt{n})\, E(\sqrt{\tau})$.

Examples show that all convergence rates in (4) are optimal in the following sense:

If $\tau_n = n\tau$—whence (2) is fulfilled for each sequence $\varepsilon_n$— you cannot obtain a better approximation order than $O(n^{-\alpha}(\lg n)^{\alpha+\beta})$ for $H_n(\tau)$ under assumption (3). If $\tau$ is a constant limit function—whence $\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) \equiv 0$—you cannot obtain a better approximation order than $O(\varepsilon_n^{1/2})$ for $H_n(\tau)$ under assumption (2).

Our Example 2.6 explains the occurrence of the special sequence $n^{-\alpha}(\lg n)^{\beta}$ in Theorem 2.1.

Applications of Theorem 2.1 yield:

(a) If $\tau_n$ are stopping times, $\tau(\Omega)$ is finite, $0<\alpha<\frac{1}{2}$, and

$$P\left\{\left|\frac{\tau_n}{n\tau}-1\right|>n^{-2\alpha}\right\} = O((n \lg n)^{-\alpha}),$$

then

$$\sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_{\tau_n}}{\tau_n^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right| = O(n^{-\alpha}).$$

(b) If $\tau: \Omega \to \mathbb{N}$ is a stopping time with $E(\tau^{\delta})<\infty$ for some $\delta>\frac{1}{2}$, then

$$\sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_{n\tau}}{(n\tau)^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right| = O(n^{-1/2}).$$

Part (a) follows from Corollary 2.12, applied to $\beta = -\alpha$. Part (b) follows from Theorem 2.1 applied to $\tau_n = n\tau$, $\varepsilon_n = 1/n$, $\alpha = \frac{1}{2}$, and $\beta = -2$: Obviously (2.2) holds; (2.3) holds as

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) \leqslant P\{\tau > n\} \leqslant (1/n^\delta \, E(\tau^\delta)) = O(n^{-1/2}(\lg n)^{-2}).$$

Results on convergence rates in the random central limit theorem for the special case that the random indices $\tau_n$ are independent from the process $X_n$, $n \in \mathbb{N}$, can be found in [7–9, 13, 25]; in the first two papers $X_n$ is even a martingale difference sequence.

## 2. The Results

The following theorem is the main result of this paper. The proof is given in Section 3.

2.1. Theorem. *Let* $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, *be* i.i.d. *with positive definite covariance matrix* $V$. *Let* $\tau_n : \Omega \to \mathbb{N}$, $n \in \mathbb{N}$, *and* $\tau : \Omega \to [c, \infty)$ *be* $\mathscr{A}$*-measurable with* $c > 0$. *Let* $0 < \varepsilon_n \to 0$, $0 < \alpha \leqslant \frac{1}{2}$, $\beta \in \mathbb{R}$, *and assume that*

$$P\left\{\left|\frac{\tau_n}{n\tau} - 1\right| > \varepsilon_n\right\} = O(\varepsilon_n^{1/2}), \tag{2.2}$$

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = O(n^{-\alpha}(\lg n)^\beta). \tag{2.3}$$

*Then*

$$\sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_{\tau_n}}{(n\tau)^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right| = O(\varepsilon_n^{1/2}) + O(\delta_n), \tag{2.4}$$

$$\sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_{\tau_n}}{\tau_n^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right| = O(\varepsilon_n^{1/2}) + O(\delta_n), \tag{2.5}$$

*where*

$$\delta_n = \delta_n(\alpha, \beta) = \begin{cases} n^{-1/2}; & \alpha = \frac{1}{2}, \ \beta < -3/2 \\ n^{-1/2} \lg \lg n; & \alpha = \frac{1}{2}, \ \beta = -\frac{3}{2} \\ n^{-1/2}(\lg n)^{\beta + 3/2}; & \alpha = \frac{1}{2}, \ \beta > -\frac{3}{2} \\ n^{-\alpha}(\lg n)^{\beta + \alpha}; & 0 < \alpha < \frac{1}{2}, \ \beta \in \mathbb{R}. \end{cases}$$

The reader might wonder why we use in (2.3) the special sequences $n^{-\alpha}(\lg n)^\beta$ and why we do not try to construct a general function $\varphi$ (e.g., $\varphi(x) = x^\gamma$) such that condition (2.2) of Theorem 2.1 and

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = O(a_n)$$

imply

$$\sup_{C \in \mathscr{C}} \left| P\left\{ \frac{S_{\tau_n}}{\tau_n^{1/2}} \in C \right\} - \Phi_{O,I}(C) \right| = O(\varepsilon_n^{1/2}) + O(\varphi(a_n)).$$

Unfortunately a result of this type does not hold as the following example shows. Observe that in this example $\tau(\Omega) = \{1, 2\}$ and $\tau_n = n\tau = [n\tau]$, whence condition (2.2) is fulfilled for each sequence $\varepsilon_n$ and therefore especially for $\varepsilon_n = 1/n$.

2.6. EXAMPLE. Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R})$, $n \in \mathbb{N}$, be i.i.d. with $E(X_1) = 0$, $E(X_1^2) = 1$ such that $P_{X_1}$ is non-atomic. Let $\varphi$: $[0, 1] \to \mathbb{R}$ be strictly increasing and continuous with $\varphi(0) = 0$. Then there exists a sequence $a_n \downarrow 0$ and a measurable function $\tau$: $\Omega \to \{1, 2\}$ such that

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = O(a_n) \tag{2.7}$$

and

$$|P\{S_{n\tau} \leqslant 0\} - \Phi(0)| \geqslant c(1/n^{1/2} + \varphi(a_n)) \tag{2.8}$$

infinitely often for each $c > 0$.

*Proof.* See Section 3.

Let us point out now that the convergence orders in Theorem 2.1 are optimal. Example 3 of [13] shows that if $\tau$ is a constant limit function (whence $\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = 0$), condition (2.2) does not guarantee a better convergence order in (2.4) and (2.5) than $O(\varepsilon_n^{1/2})$. The following example shows that condition (2.3) does not guarantee a better convergence order in (2.4) and (2.5) than $\delta_n(\alpha, \beta)$, even if $\tau(\Omega) = \{1, 2\}$ and $\tau_n = n\tau$ (whence condition (2.2) is fulfilled for each sequence $\varepsilon_n$).

2.9. EXAMPLE. Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R})$, $n \in \mathbb{N}$, be i.i.d. with $E(X_1) = 0$, $E(X_1^2) = 1$ such that $P_{X_1}$ is non-atomic. Then there exists a measurable function $\tau$: $\Omega \to \{1, 2\}$ such that

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = O(n^{-\alpha}(\lg n)^{\beta}) \tag{2.10}$$

and

$$|P\{S_{n\tau} \leqslant 0\} - \Phi(0)| \geqslant \begin{cases} cn^{-1/2} \lg \lg n; & \alpha = \frac{1}{2}, \beta = -\frac{3}{2} \\ cn^{-1/2}(\lg n)^{\beta + 3/2}; & \alpha = \frac{1}{2}, \beta > -\frac{3}{2} \\ cn^{-\alpha}(\lg n)^{\beta + \alpha}; & 0 < \alpha < \frac{1}{2}, \beta \in \mathbb{R} \end{cases} \tag{2.11}$$

for infinitely many $n \in \mathbb{N}$ (where $c = c(\alpha, \beta, P_{X_1}) > 0$).

*Proof.* See Section 3.

The following result is an application of Theorem 2.1 to the case where the random summation indices $\tau_n$ are stopping times and the limit function $\tau$ assumes only finitely many values. In this case, condition (2.3) of Theorem 2.1 can be deduced from a suitable form of condition (2.2).

2.12. COROLLARY. *Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, be* i.i.d. *with positive definite covariance matrix $V$. Let $\tau_n: \Omega \to \mathbb{N}$, $n \in \mathbb{N}$, be stopping times and $\tau$: $\Omega \to (0, \infty)$ be $\mathscr{A}$-measurable such that $\tau(\Omega)$ is finite. Let $0 < \alpha \leqslant \frac{1}{2}$, $\beta \in \mathbb{R}$ and let $\delta_n = \delta_n(\alpha, \beta)$ be defined as in Theorem 2.1. Assume that*

$$(*) \qquad P\left\{\left|\frac{\tau_n}{n\tau} - 1\right| > \delta_n^2\right\} = O(n^{-\alpha}(\lg n)^\beta).$$

*Then*

$$\text{(a)} \qquad \sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_{\tau_n}}{(n\tau)^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right| = O(\delta_n),$$

$$\text{(b)} \qquad \sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_{\tau_n}}{\tau_n^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right| = O(\delta_n).$$

*Proof.* Since $n^{-\alpha}(\lg n)^\beta = O(\delta_n)$, assumption (2.2) of Theorem 2.1 is fulfilled with $\varepsilon_n = \delta_n^2$ according to $(*)$. Hence the assertion follows from Theorem 2.1 if we show that

$$(1) \qquad \rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = O(n^{-\alpha}(\lg n)^\beta).$$

Since $\tau(\Omega)$ is finite, (1) is shown if we prove for each $b \in \tau(\Omega)$ that

$$(2) \qquad d(\{\tau = b\}, \sigma(X_1, ..., X_n)) = O(n^{-\alpha}(\lg n)^\beta).$$

Put

$$A(n, b) = \left\{\left|\frac{\tau_n}{nb} - 1\right| \leqslant \delta_n^2\right\}, \qquad n \in \mathbb{N},\ b \in \tau(\Omega).$$

Since $\tau(\Omega)$ is finite there exists $n_0 \in \mathbb{N}$ such that

(3) $\quad A(n, b)$, $b \in \tau(\Omega)$, are disjoint for all $n \geqslant n_0$.

Let $b \in \tau(\Omega)$ be fixed and put $k(n) := \max\{j \in \mathbb{N}: j \leqslant bn(1 + \delta_n^2)\}$. Since $\tau_n$, $n \in \mathbb{N}$, are stopping times, we have

$$(4) \qquad A(n, b) \in \sigma(X_1, ..., X_{k(n)}).$$

By (3) we obtain for all $n \geqslant n_0$ that

(5) $$\{\tau = b\} \,\Delta A(n, b) \subset \left\{ \left| \frac{\tau_n}{n\tau} - 1 \right| > \delta_n^2 \right\}.$$

Hence (4), (5), and (*) imply

(6) $$d(\{\tau = b\}, \sigma(X_1, ..., X_{k(n)})) = O(n^{-\alpha}(\lg n)^{\beta}).$$

Since $k(n) \leqslant 2bn$ for sufficiently large $n \in \mathbb{N}$, (6) implies (2).

*Remarks.* (a) It is possible to prove modified versions of Theorem 2.1 and Corollary 2.12 under a weaker moment condition ($X_1 \in \mathscr{L}_{2+\varepsilon}$ for some $0 < \varepsilon < 1$), using

$$\sup_{C \in \mathscr{C}} \left| P\left\{ \frac{S_n}{n^{1/2}} \in C \right\} - \Phi_{O,I}(C) \right| \leqslant cn^{-\varepsilon/2}$$

(see formula (18.25) of [3]) instead of $\leqslant cn^{-1/2}$.

(b) If we replace condition (2.3) of Theorem 2.1 by

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n, Y)) = O(n^{-\alpha}(\lg n)^{\beta}),$$

where $Y$ is independent of $X_n$, $n \in \mathbb{N}$, we obtain a slight generalization of Theorem 2.1. The proof does not change. This generalization essentially contains a result of [14], where $\tau$ is independent of $X_n$, $n \in \mathbb{N}$; choose $Y = \tau$.

## 3. Proof of the Results

In this section we prove the results of Section 2, postponing the proofs of some auxiliary lemmata to Section 4.

Put $[x] = \min\{n \in \mathbb{N} : x \leqslant n\}$ for $x \in \mathbb{R}$.

*Proof of Theorem* 2.1. Let w.l.g. $E(X_1) = 0$, $V = I$. As $\delta_n \geqslant n^{-1/2}$, we may w.l.g. assume that $\varepsilon_n \geqslant 1/(cn)$. Hence (2.2) implies

$$P\left\{ \left| \frac{\tau_n}{[n\tau]} - 1 \right| > 2\varepsilon_n \right\} = O(\varepsilon_n^{1/2}).$$

Considering $\hat{\varepsilon}_n = 2\varepsilon_n$ instead of $\varepsilon_n$ we may therefore replace condition (2.2) by

[i] $$P\left\{ \left| \frac{\tau_n}{[n\tau]} - 1 \right| > \varepsilon_n \right\} = O(\varepsilon_n^{1/2}) \qquad \text{with} \quad \varepsilon_n \geqslant \frac{1}{cn}.$$

We will show that

$$\text{(I)} \qquad \sup_{C \in \mathscr{C}} \left| P\left\{ \frac{S_{[n\tau]}}{[n\tau]^{1/2}} \in C \right\} - \Phi_{O,I}(C) \right| = O(\delta_n),$$

$$\text{(II)} \qquad \sup_{C \in \mathscr{C}} \left( P\left\{ \exists \nu \in I_n(\omega): \frac{S_\nu(\omega)}{[n\tau(\omega)]^{1/2}} \in C \right\} - P\left\{ \forall \nu \in I_n(\omega): \frac{S_\nu(\omega)}{[n\tau(\omega)]^{1/2}} \in C \right\} \right) = O(\varepsilon_n^{1/2}) + O(\delta_n)$$

where $I_n(\omega) = \{\nu \in \mathbb{N}: [n\tau(\omega)](1-\varepsilon_n) \leqslant \nu \leqslant [n\tau(\omega)](1+\varepsilon_n)\}$.

Let us at first prove that (I) and (II) imply the assertion. Put

$$A_n(C) = \left\{ \frac{S_\nu(\omega)}{[n\tau(\omega)]^{1/2}} \in C \text{ for all } \nu \in I_n(\omega) \right\},$$

$$B_n(C) = \left\{ \frac{S_\nu(\omega)}{[n\tau(\omega)]^{1/2}} \in C \text{ for some } \nu \in I_n(\omega) \right\}.$$

Since $P\{\omega: \tau_n(\omega) \notin I_n(\omega)\} = O(\varepsilon_n^{1/2})$ by [i] and since $[n\tau(\omega)] \in I_n(\omega)$, we have

$$P(A_n(C)) - O(\varepsilon_n^{1/2}) \leqslant P\left\{ \frac{S_{\tau_n}}{[n\tau]^{1/2}} \in C \right\},$$

$$P\left\{ \frac{S_{[n\tau]}}{[n\tau]^{1/2}} \in C \right\} \leqslant P(B_n(C)) + O(\varepsilon_n^{1/2}).$$

Hence (I) and (II) imply

$$\text{(1)} \qquad \sup_{C \in \mathscr{C}} \left| P\left\{ \frac{S_{\tau_n}}{[n\tau]^{1/2}} \in C \right\} - \Phi_{O,I}(C) \right| = O(\varepsilon_n^{1/2}) + O(\delta_n).$$

We have

$$\text{(2)} \qquad \left\{ \left| \frac{[n\tau]^{1/2}}{(n\tau)^{1/2}} - 1 \right| > \varepsilon_n^{1/2} \right\} \subset \left\{ \left| \frac{[n\tau]^{1/2}}{(n\tau)^{1/2}} - 1 \right| > \frac{1}{(cn)^{1/2}} \right\} \subset \left\{ \left| \frac{[n\tau]}{n\tau} - 1 \right| > \frac{1}{cn} \right\} = \varnothing.$$

We obtain (2.4) by (1), (2), and Lemma 4.4 applied to $Y_n = S_{\tau_n}/[n\tau]^{1/2}$, $\xi_n = [n\tau]^{1/2}/(n\tau)^{1/2}$, and $a_n = \varepsilon_n^{1/2} + \delta_n$. Furthermore, we have for all $n \in \mathbb{N}$ with $\varepsilon_n \leqslant \frac{1}{2}$ that

$$\left\{ \left| \frac{[n\tau]^{1/2}}{\tau_n^{1/2}} - 1 \right| > (2\varepsilon_n)^{1/2} \right\} \subset \left\{ \left| \frac{[n\tau]}{\tau_n} - 1 \right| > 2\varepsilon_n \right\} \subset \left\{ \left| \frac{\tau_n}{[n\tau]} - 1 \right| > \varepsilon_n \right\}.$$

Hence [i] implies with $a_n = (2\varepsilon_n)^{1/2} + \delta_n$ that

$$(3) \qquad P\left\{\left|\frac{[n\tau]^{1/2}}{\tau_n^{1/2}} - 1\right| > a_n\right\} = O(\varepsilon_n^{1/2}) = O(a_n).$$

Now (1), (3), and Lemma 4.4 yield (2.5).

Thus it remains to prove (I) and (II).

In the following $c_i$ are constants only depending on the distribution of $X_1$, $\alpha$, $\beta$, and the lower bound $c$ of $\tau$.

*Proof of* (I). Let $\mathbb{N}_1 = \{2^i : i \in \mathbb{N}\}$, $N_n = \{\nu \in \mathbb{N}_1 : \nu \leqslant [n/\lg n]\}$, and $j(n) = \max N_n$; $n \geqslant 3$. For each $B \in \mathscr{A}$ put

$$(4) \qquad B(\nu) := \{P(B \mid \mathscr{A}_\nu) > \tfrac{1}{2}\}, \qquad \nu \in \mathbb{N},$$

where $\mathscr{A}_\nu := \sigma(X_1, ..., X_\nu)$. Put furthermore

$$(5) \qquad A_\nu := \{|S_\nu| > \rho_3^{1/3}(2k\nu \lg \nu)^{1/2}\},$$

where $\rho_3 = E(|X_1|^3)$ and $k$ is the dimension of $\mathbb{R}^k$. We prove later that for all $B \in \mathscr{A}$, $m \in \mathbb{N}$, $m \geqslant 2$,

$$(6) \qquad \sup_{C \in \mathscr{C}} \left| P\left\{\frac{S_m}{m^{1/2}} \in C, B\right\} - \Phi_{O,I}(C)\, P(B)\right|$$
$$\leqslant d(B, \mathscr{A}_{j(m)}) + \frac{c_1}{m^{1/2}}\left(P(B(1)) + \int_{B(1)} |X_1|\, dP\right)$$
$$+ \frac{c_2}{m^{1/2}} \sum_{\nu \in N_m} \left((\nu \lg \nu)^{1/2}\, d(B, \mathscr{A}_{\nu/2})\right.$$
$$\left. + \int_{A_\nu \cap (B(\nu)\, \Delta B(\nu/2))} |S_\nu|\, dP\right).$$

We show at first that (6) implies (I). Let $n \in \mathbb{N}$ be fixed with $n \geqslant 2$ and $nc \geqslant 2$. Put

$$B_m := \{[n\tau] = m\} \in \sigma(\tau), \qquad m \in \mathbb{N}.$$

Since $\tau \geqslant c > 0$ we have for all $C \in \mathscr{C}$ that

$$(7) \qquad \left| P\left\{\frac{S_{[n\tau]}}{[n\tau]^{1/2}} \in C\right\} - \Phi_{O,I}(C)\right|$$
$$= \left| \sum_{m \geqslant nc} P\left\{\frac{S_m}{m^{1/2}} \in C, [n\tau] = m\right\} - \Phi_{O,I}(C)\, P\{[n\tau] = m\}\right|$$
$$\leqslant \sum_{m \geqslant nc} \left| P\left\{\frac{S_m}{m^{1/2}} \in C, B_m\right\} - \Phi_{O,I}(C)\, P(B_m)\right|.$$

By (7) and (6)—applied to $B=B_m$ for each $m \geqslant nc$—we have

$$\left| P\left\{ \frac{S_{[n\tau]}}{[n\tau]^{1/2}} \in C \right\} - \Phi_{O,I}(C) \right| \leqslant \sum_{j=1}^{4} R_j(C, n)$$

with

$$R_1(n) = \sum_{m \geqslant nc} d(B_m, \mathscr{A}_{j(m)}),$$

$$R_2(n) = c_1 \sum_{m \geqslant nc} \frac{1}{m^{1/2}} \left( P(B_m(1)) + \int_{B_m(1)} |X_1| \, dP \right),$$

$$R_3(n) = c_2 \sum_{m \geqslant nc,\, v \in N_m} \frac{1}{m^{1/2}} (v \lg v)^{1/2} d(B_m, \mathscr{A}_{v/2}),$$

$$R_4(n) = c_2 \sum_{m \geqslant nc,\, v \in N_m} \frac{1}{m^{1/2}} \int_{A_v \cap (B_m(v) \Delta B_m(v/2))} |S_v| \, dP.$$

To prove that (6) implies (I), we have to show that

$$\sup_{C \in \mathscr{C}} R_j(n) \leqslant c_3 \delta_n, \qquad j = 1, 2, 3, 4. \tag{8}$$

As $B_m \in \sigma(\tau)$, $m \in \mathbb{N}$, are disjoint we obtain from Lemma 4.8(i) and assumption (2.3) that

$$\begin{aligned} R_1(n) &\leqslant \sum_{m \geqslant nc} d(B_m, \mathscr{A}_{j([nc])}) \\ &\leqslant 4\rho(\sigma(\tau), \mathscr{A}_{j([nc])}) \\ &\underset{(2.3)}{\leqslant} c_4 (j([nc]))^{-\alpha} (\lg j([nc]))^{\beta} \\ &\leqslant c_5 \left( \frac{n}{\lg n} \right)^{-\alpha} \left( \lg \left( \frac{n}{\lg n} \right) \right)^{\beta} \leqslant c_6 \delta_n. \end{aligned}$$

Since $B_m(1)$, $m \in \mathbb{N}$, are disjoint according to (4), we have

$$R_2(n) \leqslant c_7 / n^{1/2} \leqslant c_7 \delta_n.$$

Furthermore, we have

$$\begin{aligned} R_3(n) &= c_2 \sum_{\mu \in \mathbb{N}_1} \sum_{(\mu/2)nc \leqslant m < \mu nc,\, v \in N_m} \frac{1}{m^{1/2}} (v \lg v)^{1/2} d(B_m, \mathscr{A}_{v/2}) \\ &\leqslant c_8 \sum_{\mu \in \mathbb{N}_1} \frac{1}{(\mu n)^{1/2}} \sum_{(\mu/2)nc \leqslant m < \mu nc,\, v \in N_{[\mu nc]}} (v \lg v)^{1/2} d(B_m, \mathscr{A}_{v/2}). \end{aligned}$$

As $B_m \in \sigma(\tau)$, $m \in \mathbb{N}$, are disjoint, we obtain from Lemma 4.8(i) and assumption (2.3) that

$$R_3(n) \leqslant 4c_8 \sum_{\mu \in \mathbb{N}_1} \frac{1}{(\mu n)^{1/2}} \sum_{\nu \in N_{[\mu n c]}} (\nu \lg \nu)^{1/2} \rho(\sigma(\tau), \mathscr{A}_{\nu/2})$$
$$\underset{(2.3)}{\leqslant} c_9 \sum_{\mu \in \mathbb{N}_1} \frac{1}{(\mu n)^{1/2}} \sum_{\nu \in N_{[\mu n c]}} \nu^{1/2-\alpha} (\lg \nu)^{\beta+1/2}.$$

As $(1/p^{1/2}) \sum_{\nu \in N_p} \nu^{1/2-\alpha} (\lg \nu)^{\beta+1/2} \leqslant c_{10}\, \delta_p$ by Lemma 4.7, we obtain

$$R_3(n) \leqslant c_{11} \sum_{\mu \in \mathbb{N}_1} \delta_{[\mu n c]} \leqslant c_{12}\, \delta_n,$$

where the last inequality follows by a direct computation from the definition of $\delta_n = \delta_n(\alpha, \beta)$.

As $B_m(\nu)$, $m \in \mathbb{N}$, are disjoint for each $\nu \in \mathbb{N}$, we have

$$R_4(n) \leqslant \frac{c_{13}}{n^{1/2}} \sum_{\nu \in \mathbb{N}_1} \sum_{m \geqslant nc} \left( \int_{A_\nu \cap B_m(\nu)} |S_\nu| \, dP + \int_{A_\nu \cap B_m(\nu/2)} |S_\nu| \, dP \right)$$
$$\leqslant 2c_{13} \frac{1}{n^{1/2}} \sum_{\nu \in \mathbb{N}_1} \int_{A_\nu} |S_\nu| \, dP.$$

By Lemma 4.9 this implies

$$R_4(n) \leqslant c_{14} \frac{1}{n^{1/2}} \leqslant c_{14}\, \delta_n.$$

Hence (8) is proven. Thus (6) implies (I) and it remains to prove (6).

Let $B \in \mathscr{A}$ and $2 \leqslant m \in \mathbb{N}$ be given. Then

$$1_B = 1_B - 1_{B(j(m))} + \sum_{\nu \in N_m} (1_{B(\nu)} - 1_{B(\nu/2)}) + 1_{B(1)}. \tag{9}$$

For $\nu \in N_m \cup \{1\}$ put

$$\gamma_\nu := \sup_{C \in \mathscr{C}} |E((1_{\{S_m/m^{1/2} \in C\}} - \Phi_{O,I}(C))(1_{B(\nu)} - 1_{B(\nu/2)}))|, \tag{10}$$

where $B(\frac{1}{2}) := \phi$. By (9) and (10) we have

$$\eta_m := \sup_{C \in \mathscr{C}} \left| P\left\{ \frac{S_m}{m^{1/2}} \in C, B \right\} - \Phi_{O,I}(C)\, P(B) \right| \tag{11}$$
$$\leqslant E[|1_B - 1_{B(j(m))}|] + \sum_{\nu \in N_m \cup \{1\}} \gamma_\nu.$$

By (4) and Lemma 4.8(ii) we have

(12) $$E[|1_B - 1_{B(\nu)}|] = d(B, \mathscr{A}_\nu).$$

Hence (11) implies

(13) $$\eta_m \leqslant d(B, \mathscr{A}_{j(m)}) + \sum_{\nu \in N_m \cup \{1\}} \gamma_\nu .$$

By Lemma 4.10 we have for all $\nu \in N_m \cup \{1\}$ that

(14) $$\sup_{C \in \mathscr{C}} \left| P\left(\frac{S_m}{m^{1/2}} \in C \mid \mathscr{A}_\nu\right) - \Phi_{O,I}(C) \right| \leqslant \frac{c_{15}}{m^{1/2}} (\nu^{1/2} + |S_\nu|).$$

By (10) and (14) we have

$$\gamma_\nu = \sup_{C \in \mathscr{C}} \left| \int \left( P\left(\frac{S_m}{m^{1/2}} \in C \mid \mathscr{A}_\nu\right) - \Phi_{O,I}(C) \right) (1_{B(\nu)} - 1_{B(\nu/2)})\, dP \right|$$
$$\leqslant \frac{c_{15}}{m^{1/2}} \int (\nu^{1/2} + |S_\nu|)\, |1_{B(\nu)} - 1_{B(\nu/2)}|\, dP.$$

Hence

(15) $$\gamma_1 \leqslant \frac{c_1}{m^{1/2}} \left( P(B(1)) + \int_{B(1)} |X_1|\, dP \right)$$

and for $\nu \in N_m$

(16) $$\gamma_\nu \leqslant \frac{c_{15}}{m^{1/2}} \nu^{1/2} P(B(\nu)\, \Delta B(\nu/2))$$
$$+ \frac{c_{15}}{m^{1/2}} \int_{A_\nu \cap (B(\nu)\, \Delta B(\nu/2))} |S_\nu|\, dP$$
$$+ \frac{c_{16}}{m^{1/2}} (\nu \lg \nu)^{1/2} P(B(\nu)\, \Delta B(\nu/2)).$$

Since $P(B(\nu)\, \Delta B(\nu/2)) \leqslant 2d(B, \mathscr{A}_{\nu/2})$ by (12), we obtain from (16) for each $\nu \in N_m$ that

(17) $$\gamma_\nu \leqslant c_2 \frac{1}{m^{1/2}} \left( (\nu \lg \nu)^{1/2} d(B, \mathscr{A}_{\nu/2}) + \int_{A_\nu \cap (B(\nu)\, \Delta B(\nu/2))} |S_\nu|\, dP \right).$$

Now (13), (15), and (17) imply (6). Thus (I) is shown. It remains to prove (II).

*Proof of* (II). We have to show

$$\sup_{C \in \mathscr{C}} P(B_n(C)) - P(A_n(C)) = O(\varepsilon_n^{1/2}) + O(\delta_n). \tag{18}$$

Let $n$ be fixed and put $B_m = \{[n\tau] = m\} \in \sigma(\tau)$.

Let $I_n(m) = \{v \in \mathbb{N}: m(1-\varepsilon_n) \leqslant v \leqslant m(1+\varepsilon_n)\}$; we have

$$\begin{aligned} &P(B_n(C)) - P(A_n(C)) \\ &\quad = \sum_{m \geqslant nc} \{P(B_n(C) \cap B_m) - P(A_n(C) \cap B_m)\} \\ &\quad = \sum_{m \geqslant nc} \{P\{B_m, S_v \in m^{1/2}C \text{ for some } v \in I_n(m)\} \\ &\qquad - P\{B_m, S_v \in m^{1/2}C \text{ for all } v \in I_n(m)\}\}. \end{aligned} \tag{19}$$

Let $A_m = \{P(B_m \mid \mathscr{A}_{j(n)}) > \frac{1}{2}\}$. Then $A_m \in \mathscr{A}_{j(n)}$, $m \in \mathbb{N}$, are disjoint and $P(B_m \Delta A_m) = d(B_m, \mathscr{A}_{j(n)})$. Hence (19) implies

$$\begin{aligned} P(B_n(C)) - P(A_n(C)) \leqslant 2 &\sum_{m \geqslant nc} d(B_m, \mathscr{A}_{j(n)}) \\ &+ \sum_{m \geqslant nc} \{P\{A_m, S_v \in m^{1/2}C \text{ for some } v \in I_n(m)\} \\ &- P\{A_m, S_v \in m^{1/2}C \text{ for all } v \in I_n(m)\}\}. \end{aligned} \tag{20}$$

As $B_m$, $m \in \mathbb{N}$, are disjoint, we have by Lemma 4.8(i) and assumption (2.3) that

$$\sum_{m \geqslant nc} d(B_m, \mathscr{A}_{j(n)}) \leqslant 4\rho(\sigma(\tau), \mathscr{A}_{j(n)}) \leqslant c_{17}\, \delta_n. \tag{21}$$

Using Lemma 4.2(ii) and $A_m \in \mathscr{A}_{j(n)}$ we have for all $m \geqslant nc$ that

$$\begin{aligned} &P\{A_m, S_v \in m^{1/2}C \text{ for some } v \in I_n(m)\} \\ &\qquad - P\{A_m, S_v \in m^{1/2}C \text{ for all } v \in I_n(m)\} \\ &\quad = \int_{A_m} P(\exists v, \mu \in I_n(m): S_v \in m^{1/2}C, S_\mu \notin m^{1/2}C \mid \mathscr{A}_{j(n)})\, dP \\ &\quad \leqslant c_{18} P(A_m) \sqrt{\frac{2m\varepsilon_n}{m(1-\varepsilon_n) - j(n)}} \leqslant c_{19} P(A_m)\, \varepsilon_n^{1/2} \end{aligned} \tag{22}$$

for sufficiently large $n$. Now (20), (21), and (22) imply (18), i.e., (II) is shown.

*Construction of Example* 2.6. Let w.l.g. $\varphi(x) \geqslant x$, otherwise consider $\varphi(x) \vee x$. From the central limit theorem we directly obtain that

$$\lim_{n \to \infty} P\{S_n > 0, S_{2n} \leqslant 0\} =: b > 0. \tag{1}$$

Let $\eta_n := n^{-1/2}$ and $\psi := \varphi^{1/2}$. There exists a subsequence $i(\nu) \in \mathbb{N}$, $\nu \in \mathbb{N}$, with

$$\sum_{\nu > m} \eta_{i(\nu)} \leqslant \psi^{-1}(\eta_{i(m)}), \qquad m \in \mathbb{N}, \tag{2}$$

$$\sum_{\nu \in \mathbb{N}} \eta_{i(\nu)} \leqslant \frac{1}{4} b. \tag{3}$$

Now we inductively construct $k(m) \in \mathbb{N}$, $k(m) > k(m-1)$, and sets $B_{k(m)} \subset \Omega$ such that

$$k(m) \geqslant i(m+1), \qquad B_{k(m)} \in \sigma(X_\nu : \nu \leqslant k(m)), \tag{4}$$

$$B_{k(m)} \subset \{S_{k(m)} > 0\} - (B_{k(1)} \cup \cdots \cup B_{k(m-1)}), \tag{5}$$

$$P(B_{k(m)}) = \eta_{i(m)}, \tag{6}$$

$$\left| P\left\{ S_{k(m)} \leqslant 0, \sum_{\nu=1}^{m} B_{k(\nu)} \right\} - P\left\{ S_{2k(m)} \leqslant 0, \sum_{\nu=1}^{m} B_{k(\nu)} \right\} \right| \geqslant \frac{b}{8} \eta_{i(m)}. \tag{7}$$

Let us at first show that this construction implies the assertion. Let $B = \sum_{\nu \in \mathbb{N}} B_{k(\nu)}$ and put

$$\tau = 1_B + 2\, 1_{\Omega - B}.$$

Define the sequence $a_n$ by

$$a_n = \psi^{-1}(\eta_{i(m)}) \qquad \text{for } k(m) \leqslant n < k(m+1),\ m \in \mathbb{N}. \tag{8}$$

Let $m$ be such that $k(m) \leqslant n < k(m+1)$. Using (2) and (6) we have

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = d(B, \sigma(X_1, ..., X_n)) \leqslant \sum_{\nu > m} P(B_{k(\nu)}) \tag{9}$$

$$\underset{(6)}{=} \sum_{\nu > m} \eta_{i(\nu)} \underset{(2)}{\leqslant} \psi^{-1}(\eta_{i(m)}) = a_n.$$

Relation (9) implies (2.7). Furthermore, we obtain for all $n = k(m)$, $m \in \mathbb{N}$—using the theorem of Berry and Esseen—that

$$\begin{aligned}
&|P\{S_{n\tau}\leqslant 0\}-\Phi(0)| \\
&\quad =|P\{S_n\leqslant 0,\, B\}+P\{S_{2n}\leqslant 0,\, \Omega-B\}-\Phi(0)| \\
&\quad =|P\{S_n\leqslant 0,\, B\}-P\{S_{2n}\leqslant 0,\, B\}+P\{S_{2n}\leqslant 0\}-\Phi(0)| \\
&\quad \geqslant\left|P\left\{S_{k(m)}\leqslant 0,\, \sum_{v=1}^{m} B_{k(v)}\right\}-P\left\{S_{2k(m)}\leqslant 0,\, \sum_{v=1}^{m} B_{k(v)}\right\}\right. \\
&\qquad \left. -\sum_{v>m} P(B_{k(v)})-\frac{c_1}{n^{1/2}}\right| \\
&\quad \underset{(2),(6),(7)}{\geqslant} \frac{b}{8}\,\eta_{i(m)}-a_n-\frac{c_1}{n^{1/2}} \\
&\quad =\frac{b}{8}\,\psi(a_n)-a_n-\frac{c_1}{n^{1/2}}.
\end{aligned}$$

Since $\psi(a_n)=(\varphi(a_n))^{1/2}\geqslant a_n^{1/2}$ we consequently obtain for all $n=k(m)$ with sufficiently large $m$ that

$$P\{S_{n\tau}\leqslant 0\}-\Phi(0)\geqslant c_3(\varphi(a_n))^{1/2}-c_1/n^{1/2}.$$

Since, furthermore, for all $n=k(m)$

$$\varphi(a_n)\underset{(8)}{=}\eta_{i(m)}\underset{(4)}{\geqslant}\eta_{k(m)}=\eta_n=1/n^{1/2},$$

we obtain (2.8).

Thus it remains to construct $k(m)\in\mathbb{N}$, $B_{k(m)}\subset\Omega$ fulfilling (4)–(7).

According to (1) there exists $k(1)\geqslant i(2)$ such that

$$P\{S_{k(1)}>0,\, S_{2k(1)}\leqslant 0\}\geqslant b/2.$$

Now apply Lemma 4.6 with $\mathscr{A}_0=\sigma(X_v\colon v\leqslant k(1))$, $A=\{S_{2k(1)}\leqslant 0\}$, $A_0=\{S_{k(1)}>0\}$, $\alpha=b/2$, and $\varepsilon=\eta_{i(1)}$; then $\varepsilon\leqslant\alpha/2$ by (3) and $P(A\cap A_0)\geqslant\alpha$. Hence there exists $B_{k(1)}\subset\{S_{k(1)}>0\}$, $B_{k(1)}\in\sigma(X_1, ..., X_{k(1)})$ such that

$$P(B_{k(1)})=\eta_{i(1)}, \qquad P\{S_{2k(1)}\leqslant 0,\, B_{k(1)}\}\leqslant (b/4)\,\eta_{i(1)}.$$

Hence (4)–(6) are fulfilled for $k(1)$, $B_{k(1)}$ and (7) holds as

$$\begin{aligned}
&|P\{S_{k(1)}\leqslant 0,\, B_{k(1)}\}-P\{S_{2k(1)}\leqslant 0,\, B_{k(1)}\}| \\
&\quad =P\{S_{2k(1)}\leqslant 0,\, B_{k(1)}\}\geqslant (b/4)\,\eta_{i(1)}.
\end{aligned}$$

Now assume that $k(v)$, $B_{k(v)}$ are defined for $v\leqslant m$ such that (4)–(7) hold.

According to the conditional central limit theorem of Renyi there exists $k(m+1) \geqslant i(m+2) \vee k(m)$ such that

$$(10) \qquad \left| P\left\{ S_{k(m+1)} \leqslant 0, \sum_{v=1}^{m} B_{k(v)} \right\} - P\left\{ S_{2k(m+1)} \leqslant 0, \sum_{v=1}^{m} B_{k(v)} \right\} \right| \leqslant \frac{b}{8} \eta_{i(m+1)}.$$

Since $\sum_{v=1}^{m} P(B_{k(v)}) \leqslant b/4$ by (3) and (6), $k(m+1)$ can be chosen according to (1), such that additionally

$$(11) \qquad P\left( \{ S_{k(m+1)} > 0, S_{2k(m+1)} \leqslant 0 \} - \bigcup_{v \leqslant m} B_{k(v)} \right) \geqslant P\{ S_{k(m+1)} > 0, S_{2k(m+1)} \leqslant 0 \} - \sum_{v=1}^{m} P(B_{k(v)}) \geqslant \frac{b}{2}.$$

Now apply Lemma 4.6 with $\mathscr{A}_0 = \sigma(X_v : v \leqslant k(m+1))$, $A = \{ S_{2k(m+1)} \leqslant 0 \}$, $A_0 = \{ S_{k(m+1)} > 0 \} - \bigcup_{v \leqslant m} B_{k(v)}$, $\alpha = b/2$, and $\varepsilon = \eta_{i(m+1)}$; then $\varepsilon \leqslant \alpha/2$ by (3) and $P(A \cap A_0) \geqslant \alpha$ by (11). Hence there exists $B_{k(m+1)} \in \sigma(X_v : v \leqslant k(m+1))$ such that

$$(12) \qquad B_{k(m+1)} \subset \{ S_{k(m+1)} > 0 \} - \bigcup_{v \leqslant m} B_{k(v)},$$

$$(13) \qquad P(B_{k(m+1)}) = \eta_{i(m+1)},$$

$$(14) \qquad P\{ S_{2k(m+1)} \leqslant 0, B_{k(m+1)} \} \geqslant (b/4)\, \eta_{i(m+1)}.$$

Thus (4)–(6) are fulfilled for $m+1$. It remains to prove (7). We have

$$\left| P\left\{ S_{k(m+1)} \leqslant 0, \sum_{v=1}^{m+1} B_{k(v)} \right\} - P\left\{ S_{2k(m+1)} \leqslant 0, \sum_{v=1}^{m+1} B_{k(v)} \right\} \right|$$

$$\underset{(12)}{=} \left| P\left\{ S_{k(m+1)} \leqslant 0, \sum_{v=1}^{m} B_{k(v)} \right\} - P\left\{ S_{2k(m+1)} \leqslant 0, \sum_{v=1}^{m} B_{k(v)} \right\} - P\{ S_{2k(m+1)} \leqslant 0, B_{k(m+1)} \} \right|$$

$$\underset{(10)}{\geqslant} P\{ S_{2k(m+1)} \leqslant 0, B_{k(m+1)} \} - \frac{b}{8} \eta_{i(m+1)}$$

$$\underset{(14)}{\geqslant} \frac{b}{4} \eta_{i(m+1)} - \frac{b}{8} \eta_{i(m+1)} = \frac{b}{8} \eta_{i(m+1)}.$$

Thus (7) holds for $m+1$. This finishes the inductive construction of $k(m)$, $B_{k(m)}$.

*Construction of Example* 2.9. Let $\alpha$, $\beta$ be fixed. There exists $n_1 \in \mathbb{N}$ such that

$$(1) \qquad \varepsilon_n := n^{-\alpha}(\lg n)^\beta \text{ is decreasing and } \leqslant \tfrac{1}{2} \text{ for } n \geqslant n_1.$$

Put $\varepsilon_n = \frac{1}{2}$ for $n < n_1$. Then there exist, according to Lemma 4.5, disjoint sets $B_\nu \in \sigma(X_1, ..., X_\nu)$, $\nu \in \mathbb{N}$, such that with $B = \sum_{\nu \in \mathbb{N}} B_\nu$,

$$(2) \qquad d(B, \sigma(X_1, ..., X_n)) \leqslant \sum_{\nu \geqslant n} P(B_\nu) \leqslant \sum_{\nu \geqslant n} (\varepsilon_\nu - \varepsilon_{\nu+1}) = \varepsilon_n,$$

$$(3) \qquad P(S_{2n} \leqslant 0, B) - P(S_n \leqslant 0, B) \geqslant \frac{c_0}{n^{1/2}} \sum_{\nu=1}^{[n/\lg n]} (\nu \lg \nu)^{1/2} (\varepsilon_\nu - \varepsilon_{\nu+1}) - \varepsilon_n$$

for infinitely many $n \in \mathbb{N}$ and some $c_0 > 0$.

Put $\tau = 1_B + 2\, 1_{\Omega - B}$. Then (2) implies

$$\rho(\sigma(\tau), \sigma(X_1, ..., X_n)) = d(B, \sigma(X_1, ..., X_n)) = O(n^{-\alpha}(\lg n)^\beta),$$

i.e., (2.10) is fulfilled. Since $\varepsilon_\nu - \varepsilon_{\nu+1} \geqslant c_1(1/\nu^{\alpha+1})(\lg \nu)^\beta$ for sufficiently large $\nu$, it is easy to see that for some $n_2 > n_1$

$$(4) \qquad \frac{1}{n^{1/2}} \sum_{\nu=1}^{[n/\lg n]} (\nu \lg \nu)^{1/2} (\varepsilon_\nu - \varepsilon_{\nu+1}) \geqslant c_2\, \delta_n \qquad \text{for all} \quad n \geqslant n_2,$$

where

$$\delta_n = \delta_n(\alpha, \beta) = \begin{cases} n^{-1/2} \lg \lg n, & \alpha = \frac{1}{2},\ \beta = -\frac{3}{2} \\ n^{-1/2} (\lg n)^{\beta + 3/2}, & \alpha = \frac{1}{2},\ \beta > -\frac{3}{2} \\ n^{-\alpha} (\lg n)^{\alpha+\beta}, & 0 < \alpha < \frac{1}{2},\ \beta \in \mathbb{R}. \end{cases}$$

Now let $n \geqslant n_2$ be such that (3) holds. Then

$$\begin{aligned} \Phi(0) - P(S_{n\tau} \leqslant 0) &= \Phi(0) - (P(S_n \leqslant 0, B) + P(S_{2n} \leqslant 0, \Omega - B)) \\ &= P(S_{2n} \leqslant 0, B) - P(S_n \leqslant 0, B) + \Phi(0) - P(S_{2n} \leqslant 0) \\ &\underset{(3),(4)}{\geqslant} c_3\, \delta_n - \varepsilon_n + \Phi(0) - P(S_{2n} \leqslant 0) \end{aligned}$$

and hence by the theorem of Berry and Esseen

$$\geqslant c_3\, \delta_n - \varepsilon_n - c_4(1/n^{1/2}) \geqslant c\, \delta_n,$$

if $n$ is sufficiently large. Hence (2.11) holds for infinitely many $n \in \mathbb{N}$.

## 4. Auxiliary Lemmas

In this section we collect all lemmas which are used for the proofs of our results.

To deal with arbitrary convex sets instead of rectangles in Theorem 2.1, we need the first four lemmas.

For $C \subset \mathbb{R}^k$, $y \in \mathbb{R}^k$ put $d(y, C) = \inf_{z \in C} |y - z|$ and $K_\varepsilon(y) = \{z \in \mathbb{R}^k: |y - z| < \varepsilon\}$ for $\varepsilon > 0$. Furthermore, let

$$C^\varepsilon := \{y \in \mathbb{R}^k: d(y, C) < \varepsilon\}$$

and

$$C^{-\varepsilon} := \{y \in \mathbb{R}^k: K_\varepsilon(y) \subset C\}.$$

It is well known that $C \in \mathscr{C}$ implies $C^\varepsilon \in \mathscr{C}$, $C^{-\varepsilon} \in \mathscr{C}$.

Lemma 4.1. *For each $C \in \mathscr{C}$ we have*

(i) $\bar{C}^{-\varepsilon} = C^{-\varepsilon}$,

(ii) $(C^\varepsilon)^{-\varepsilon} \subset \bar{C}$,

(iii) $(C - z) - (C - z)^{-2|z|} \subset C^{2r} - (C^{2r})^{-5r}$ for $z \in \mathbb{R}^k$, $|z| \leqslant r$,

*where $\bar{C}$ is the closure of $C$ and $C - z = \{c - z: c \in C\}$.*

*Proof.* Part (i) follows from the fact that the interior of $\bar{C}$ is equal to the interior of $C$.

(ii) Let w.l.g. $C = \bar{C}$. We have to prove

$$y \notin C \Rightarrow K_\varepsilon(y) \cap (\mathbb{R}^k - C^\varepsilon) \neq \varnothing. \tag{1}$$

Let $y \notin C$ be given. Then there exists $y_0 \in C$ with

$$|y - y_0| = \inf_{c \in C} |y - c|.$$

Choose $f \in \mathbb{R}^k$ with

$$|f| = 1 \quad \text{and} \quad \langle f, y - c \rangle \geqslant |y - y_0| \quad \text{for all} \quad c \in C, \tag{2}$$

where $\langle x, y \rangle$ is the scalar product of $x, y \in \mathbb{R}^k$. For existence see Theorem 1.1 of [24, p. 360].

Let $z = y + (\varepsilon'/|y - y_0|)(y - y_0)$ with $\varepsilon' = \max(0, \varepsilon - |y - y_0|)$. Then we obtain, using (2), that $z \in K_\varepsilon(y) \cap (\mathbb{R}^k - C^\varepsilon)$. Hence (1) is shown.

(iii) As $C - z \subset C^{2r}$, we have to prove $(C^{2r})^{-5r} \subset (C - z)^{-2|z|}$. As $C^{-3r} \subset C^{-3|z|} \subset (C - z)^{-2|z|}$, it suffices to prove $(C^{2r})^{-5r} \subset C^{-3r}$. This follows from (i) and (ii):

$$(C^{2r})^{-5r} \subset [(C^{2r})^{-2r}]^{-3r} \underset{\text{(ii)}}{\subset} \bar{C}^{-3r} \underset{\text{(i)}}{=} C^{-3r}.$$

LEMMA 4.2. *Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k)$ be* i.i.d. *with $E(X_1)=0$ and covariance matrix I. Then there exists a constant $c_0$—depending on the distribution of $X_1$ only—such that*

(i) $$\sup_{C\in\mathscr{C}} P\{\exists v, \mu \in [p,q]: S_v \in C, S_\mu \notin C\}$$
$$\leqslant c_0 \sqrt{(q-p)/p};\ p, q \in \mathbb{N},\ p<q,$$

(ii) $$\sup_{C\in\mathscr{C}} P(\exists v, \mu \in [p,q]: S_v \in C, S_\mu \notin C \mid X_1, ..., X_j)$$
$$\leqslant c_0 \sqrt{(q-p)/(p-j)};\ j<p<q.$$

*Proof.* (i) Since $P\{\exists v, \mu \in [p,q]: S_v \in C, S_\mu \notin C\} = P\{S_p \notin C, \exists v \in (p,q]: S_v \in C\} + P\{S_p \in C, \exists \mu \in (p,q]: S_\mu \notin C\}$ it suffices to prove

$$\text{(I)}\qquad \sup_{C\in\mathscr{C}} P\{S_p \notin C, \exists v \in (p,q]: S_v \in C\} \leqslant c\sqrt{\frac{q-p}{p}},$$

$$\text{(II)}\qquad \sup_{C\in\mathscr{C}} P\{S_p \in C, \exists v \in (p,q]: S_v \notin C\} \leqslant c\sqrt{\frac{q-p}{p}}.$$

*Proof of* (I). Let $C \in \mathscr{C}$ and $p<q$ be given. Put $Y_v := X_{p+v}$, $v \in \mathbb{N}$.

$$A_{p,q} = P\{S_p \notin C, \exists v \in (p,q]: S_v \in C\}$$
$$= P\left\{S_p \notin C, S_p \in C - \sum_{j=1}^{v} Y_j \text{ for some } v \leqslant q-p\right\}.$$

As $S_p$ is independent from $Y_1, ..., Y_{q-p}$ we obtain that

$$\text{(1)}\qquad A_{p,q} = \int P\left\{S_p \notin C, S_p \in C - \sum_{j=1}^{v} y_j \text{ for some } v \leqslant q-p\right\} P_Y(dy),$$

where $Y = (Y_1, ..., Y_{q-p})$ and $y = (y_1, ..., y_{q-p})$. As $(D-z)^{-2|z|} \subset D$ for all $z \in \mathbb{R}^k$, $D \subset \mathbb{R}^k$, we have $(D-z) - D \subset (D-z) - (D-z)^{-2|z|}$, and hence with $D = (1/\sqrt{p})C$ and $z_v = (1/\sqrt{p})(y_1 + \cdots + y_v)$

$$\text{(2)}\qquad \left\{S_p \notin C, S_p \in C - \sum_{j=1}^{v} y_j \text{ for some } v \leqslant q-p\right\}$$
$$\subset \bigcup_{v=1}^{q-p} \left\{\frac{S_p}{\sqrt{p}} \notin D, \frac{S_p}{\sqrt{p}} \in D - \frac{1}{\sqrt{p}}\sum_{j=1}^{v} y_j\right\}$$
$$\subset \bigcup_{v=1}^{q-p} \left\{\frac{S_p}{\sqrt{p}} \in (D-z_v) - (D-z_v)^{-2|z_v|}\right\}.$$

By Lemma 4.1 we have

$$\bigcup_{\nu=1}^{q-p} (D - z_\nu) - (D - z_\nu)^{-2|z_\nu|} \subset D^{2a(y)} - (D^{2a(y)})^{-5a(y)},$$

where $a(y) = \max\{|z_\nu|: \quad 1 \leqslant \nu \leqslant q-p\} = \max\{|(1/\sqrt{p})(y_1 + \cdots + y_\nu)|: 1 \leqslant \nu \leqslant q-p\}$. Hence by (1) and (2)

$$A_{p,q} \leqslant \int P\left\{\frac{S_p}{\sqrt{p}} \in D^{2a(y)} - (D^{2a(y)})^{-5a(y)}\right\} P_Y(dy).$$

As $D^{2a(y)} \in \mathscr{C}$, $(D^{2a(y)})^{-5a(y)} \in \mathscr{C}$, we obtain from Corollary 17.2 of [3] that

$$A_{p,q} \leqslant \frac{c_1}{\sqrt{p}} + \int \Phi_{O,I}(D^{2a(y)} - (D^{2a(y)})^{-5a(y)})\, P_Y(dy).$$

As $\sup_{C \in \mathscr{C}} \Phi_{O,I}(C - C^{-\varepsilon}) \leqslant c_2 \varepsilon$ by Corollary 3.2 of [3], we obtain

$$\begin{aligned} A_{p,q} &\leqslant \frac{c_1}{\sqrt{p}} + c_2 \int 5a(y)\, P_Y(dy) \\ &\leqslant \frac{c_1}{\sqrt{p}} + 5c_2 \sqrt{\frac{q-p}{p}}\, E\left(\max_{\nu \leqslant q-p} \frac{|Y_1 + \cdots + Y_\nu|}{\sqrt{q-p}}\right) \leqslant c \sqrt{\frac{q-p}{p}}, \end{aligned}$$

where the last relation follows from a well-known inequality. Equation (II) runs similarly as (I) but is somewhat easier.

(ii) Put $Y_i = X_{j+i}$, $i \in \mathbb{N}$, $\hat{\mathbf{S}}_m = \sum_{i=1}^m Y_i$. As $(X_1, ..., X_j)$ and $(Y_1, ..., Y_{q-j})$ are independent we obtain that

$$P\left\{\exists \nu, \mu \in [p,q]: \hat{\mathbf{S}}_{\nu-j} \in C - \sum_{i=1}^{j} x_i, \hat{\mathbf{S}}_{\mu-j} \notin C - \sum_{i=1}^{j} x_i\right\} \tag{3}$$
$$\in P(\exists \nu, \mu \in [p,q]: S_\nu \in C, S_\mu \notin C \mid X_1, ..., X_j).$$

As

$$\begin{aligned} &P\left\{\exists \nu, \mu \in [p,q]: \hat{\mathbf{S}}_{\nu-j} \in C - \sum_{i=1}^{j} x_i, \hat{\mathbf{S}}_{\mu-j} \notin C - \sum_{i=1}^{j} x_i\right\} \\ &\quad = P\left\{\exists \nu, \mu \in [p-j, q-j]: S_\nu \in C - \sum_{i=1}^{j} x_i, S_\mu \notin C - \sum_{i=1}^{j} x_i\right\}, \end{aligned}$$

we obtain (ii) from (3) and (i).

LEMMA 4.3. *Let* $0<a\leqslant\frac{1}{2}$ *and* $C\in\mathscr{C}$. *Put*

$$\underline{C}(a)=\bigcap\left\{\eta C:\eta\in\left[\frac{1}{1+a},\frac{1}{1-a}\right]\right\},\qquad \bar{C}(a)=\bigcup_{y\in C}\left[\frac{1}{1+a}\,y,\frac{1}{1-a}\,y\right].$$

*Then* $\underline{C}(a)$, $\bar{C}(a)\in\mathscr{C}$, $\underline{C}(a)\subset C\subset\bar{C}(a)$ *and*

$$(*)\qquad \sup_{C\in\mathscr{C}}\Phi_{O,I}(\bar{C}(a)-\underline{C}(a))\leqslant c(k)\cdot a$$

*with a suitable constant* $c(k)$, *depending only on the dimension* $k$.

*Proof.* As $C\in\mathscr{C}$ we obviously have $\underline{C}(a)\in\mathscr{C}$ and

$$(1)\qquad \underline{C}(a)=\frac{C}{1+a}\cap\frac{C}{1-a}.$$

A little reflection shows also that $\bar{C}(a)\in\mathscr{C}$. We show at first that

$$(2)\qquad \sup_{C\in\mathscr{C}}\Phi_{O,I}(C-\lambda C)\leqslant c_1(k)(1-\lambda),\qquad 0<\lambda<1.$$

Let $D\in\mathscr{C}$ with $0\in D$. Then $\lambda D\subset D$ and we obtain according to Lemma 4 of [17] applied of $f=1_{\lambda D}$ and $a=\lambda$ that

$$(3)\qquad \Phi_{O,I}(D-\lambda D)=\int(1_{\lambda D}(\lambda x)-1_{\lambda D}(x))\,\Phi_{O,I}(dx)$$
$$\leqslant c_1(k)(1-\lambda).$$

Now let $\phi\neq C\in\mathscr{C}$ and put $D=\bigcup\{\eta C:0\leqslant\eta\leqslant 1\}$. It is easy to see that $0\in D\in\mathscr{C}$ and $C-\lambda C\subset D-\lambda D$. Hence (3) implies (2).

To prove $(*)$ it suffices to show that

$$(4)\qquad \sup_{C\in\mathscr{C}}\Phi_{O,I}(C-\underline{C}(a))\leqslant c_2(k)\,a,$$

$$(5)\qquad \sup_{C\in\mathscr{C}}\Phi_{O,I}(\bar{C}(a)-C)\leqslant c_3(k)a.$$

*Proof of* (4). We have by (1) and Lemma 4 of [17]—applied to $f=1_c$ and $1-a$ instead of $a$—that

$$(6)\ \Phi_{O,I}(C)-\Phi_{O,I}(\underline{C}(a))=\int(1_C(x)-1_{C/(1+a)\cap C/(1-a)}(x))\,\Phi_{O,I}(dx)$$
$$=\int(1_C(x)-1_C((1-a)\,x))\,\Phi_{O,I}(dx)$$
$$+\int(1_{C/1-a)}(x)-1_{C/(1-a)\cap C/(1+a)}(x))\,\Phi_{O,I}(dx)$$
$$\leqslant c_1(k)a+\Phi_{O,I}\left(\frac{C}{1-a}-\frac{C}{1+a}\right).$$

Put $D = C/(1-a)$. Then $C/(1+a) = ((1-a)/(1+a))D$ and hence by (2)

$$\Phi_{O,I}\left(\frac{C}{1-a} - \frac{C}{1+a}\right) = \Phi_{O,I}\left(D - \frac{1-a}{1+a}D\right) \leqslant c_1(k)\left(1 - \frac{1-a}{1+a}\right) \leqslant 2c_1(k)a. \tag{7}$$

Now (6) and (7) imply (4).

*Proof of* (5). Put $C_\lambda = \bigcup_{y \in C} [y, \lambda y]$, $\lambda > 1$. We show that

$$\sup_{C \in \mathscr{C}} \Phi_{O,I}(C_\lambda - C) \leqslant c_4(k)(\lambda - 1). \tag{8}$$

Let $D \in \mathscr{C}$ with $0 \in D$. Then $D_\lambda = \lambda D$ and we obtain as in formula (3) that

$$\Phi_{O,I}(D_\lambda - D) \leqslant c_1(k)(\lambda - 1). \tag{9}$$

If $\phi \neq C \in \mathscr{C}$, we have $D = \bigcup \{\eta C: 0 \leqslant \eta \leqslant 1\}$ that $0 \in D \in \mathscr{C}$ and $C_\lambda - C \subset D_\lambda - D$. Hence (9) implies (8). To prove (5) put $D = C/(1+a)$. Then $\bar{C}(a) = D_{(1+a)/(1-a)}$ and we obtain from (8) and Lemma 4 of [17] that

$$\begin{aligned}
&\Phi_{O,I}(\bar{C}(a) - C) \\
&\quad = \Phi_{O,I}(D_{(1+a)/(1-a)} - D) + \Phi_{O,I}(D) - \Phi_{O,I}(C) \\
&\quad \leqslant c_4(k)\left(\frac{1+a}{1-a} - 1\right) + \int \left(1_{C/(1+a)}(y) - 1_{C/(1+a)}\left(\frac{y}{1+a}\right)\right) \Phi_{O,I}(dx) \\
&\quad \leqslant 4c_4(k)\, a + c_1(k)\left(1 - \frac{1}{1+a}\right) \leqslant c_3(k)a.
\end{aligned}$$

This proves (5).

LEMMA 4.4. *Let* $0 < a_n \to 0$. Let $Y_n: \Omega \to \mathbb{R}^k$ and $\xi_n: \Omega \to \mathbb{R}$ *be random variables. Assume that*

(i) $\sup_{C \in \mathscr{C}} |P\{Y_n \in C\} - \Phi_{O,I}(C)| = O(a_n)$,

(ii) $P\{|1 - \xi_n| > a_n\} = O(a_n)$.

*Then*

$$\sup_{C \in \mathscr{C}} |P\{\xi_n Y_n \in C\} - \Phi_{O,I}(C)| = O(a_n).$$

*Proof.* Let $C \in \mathscr{C}$ and $n \in \mathbb{N}$ with $a_n \leqslant \frac{1}{2}$ be given. With $\underline{C}(a_n)$, $\bar{C}(a_n)$ of Lemma 4.3 we have

$$\{Y_n \in \underline{C}(a_n)\} \cap \{|1-\xi_n| \leqslant a_n\} \subset \{\xi_n Y_n \in C\} \cap \{|1-\xi_n| \leqslant a_n\} \subset \{Y_n \in \bar{C}(a_n)\}.$$

Hence we obtain from (ii) that

(1) $P\{Y_n \in \underline{C}(a_n)\} - O(a_n) \leqslant P\{\xi_n Y_n \in C\} \leqslant P\{Y_n \in \bar{C}(a_n)\} + O(a_n)$.

By Lemma 4.3 we have

(2) $$\sup_{C \in \mathscr{C}} \Phi_{O,I}(\bar{C}(a_n)) - \Phi_{O,I}(C) \leqslant c(k)\, a_n,$$

(3) $$\sup_{C \in \mathscr{C}} \Phi_{O,I}(C) - \Phi_{O,I}(\underline{C}(a_n)) \leqslant c(k)\, a_n.$$

As $\underline{C}(a_n)$, $\bar{C}(a_n) \in \mathscr{C}$ by Lemma 4.3, (1), (2), (3), and (i) imply the assertion.

LEMMA 4.5. *Let $X_n \in \mathscr{L}_3$, $n \in \mathbb{N}$, be* i.i.d. *with $E(X_1)=0$, $E(X_1^2)=1$ such that $P_{X_1}$ is non-atomic. Let $\varepsilon_n \downarrow$ with $\varepsilon_n = O(n^{-\gamma})$ for some $\gamma > 0$. Then there exist disjoint $B_v \in \sigma(X_1, ..., X_v)$, $v \in \mathbb{N}$, such that with $B = \sum_{v \in \mathbb{N}} B_v$,*

(1) $$P(B_v) \leqslant \varepsilon_v - \varepsilon_{v+1},$$

(2) $$P(S_{2n} \leqslant 0, B) - P(S_n \leqslant 0, B) \geqslant \frac{c}{n^{1/2}} \sum_{v=1}^{[n/\lg n]} (v \lg v)^{1/2} (\varepsilon_v - \varepsilon_{v+1}) - \varepsilon_n$$

*for infinitely many $n \in \mathbb{N}$ and some $c > 0$.*

*Proof.* The proof follows the lines of the proof of Lemma 5 of [15]. You have to replace $\Phi(0) \cdot P(B_v)$ by $P(S_{2n} \leqslant 0, B_v)$ and you have to use instead of Lemma 4 of [15] the following modified version:

For all $0 < \gamma_1 < \gamma_2$ there exist $c_0 = c_0(\gamma_1, \gamma_2) > 0$ and $n_0 = n_0(\gamma_1, \gamma_2) \in \mathbb{N}$ such that

$$P(S_{2n} \leqslant 0, B_v) - P(S_n \leqslant 0, B_v) \geqslant c_0 \left(\frac{v \lg v}{n}\right)^{1/2} P(B_v),$$

if $\sigma(X_1, ..., X_v) \ni B_v \subset \{\gamma_1 (v \lg v)^{1/2} \leqslant S_v \leqslant \gamma_2 (v \lg v)^{1/2}\}$, $v \geqslant n_0$, and $v \lg v \leqslant n$ (which is proven in a similar way as Lemma 4 of [15]).

LEMMA 4.6. *Let $P|\mathscr{A}$ be a p-measure and $\mathscr{A}_0 \subset \mathscr{A}$ a σ-field such that $P|\mathscr{A}_0$ is non-atomic. Let $A \in \mathscr{A}$, $A_0 \in \mathscr{A}_0$ be such that $P(A \cap A_0) \geqslant \alpha > 0$. Then for each $\varepsilon \leqslant \alpha/2$ there exists a set $A_\varepsilon \in \mathscr{A}_0$, $A_\varepsilon \subset A_0$ such that*

$$P(A_\varepsilon) = \varepsilon \qquad \textit{and} \qquad P(A \cap A_\varepsilon) \geqslant (\alpha/2)\varepsilon.$$

*Proof.* Let $\varepsilon \leqslant \alpha/2$ be fixed. Put $m := \max\{n \in \mathbb{N} : n\varepsilon \leqslant P(A_0)\}$. Then $m \geqslant 2$ and $P(A_0) = m\varepsilon + r$ with $r \leqslant \varepsilon$. Since $P | \mathscr{A}_0$ is non-atomic and $A_0 \in \mathscr{A}_0$ there exist—according to a theorem of Ljapunov—disjoint sets $A_1, ..., A_m \in \mathscr{A}_0$ with $A_i \subset A_0$ and $P(A_i) = \varepsilon$, $i = 1, ..., m$. Hence

$$\sum_{i=1}^{m} P(A \cap A_i) = P\left(A \cap \sum_{i=1}^{m} A_i\right) \geqslant P(A \cap A_0) - P\left(A_0 - \sum_{i=1}^{m} A_i\right)$$
$$= P(A \cap A_0) - (P(A_0) - m\varepsilon)) \geqslant \alpha - r \geqslant \alpha - \varepsilon \geqslant \alpha/2.$$

Consequently there exists $i_0 \in \{1, ..., m\}$ such that $P(A \cap A_{i_0}) \geqslant (1/m)(\alpha/2)$. Put $A_\varepsilon := A_{i_0}$. Then $A_\varepsilon \subset A_0$, $A_\varepsilon \in \mathscr{A}_0$, $P(A_\varepsilon) = \varepsilon$, and—as $m \leqslant (1/\varepsilon)\, P(A_0)$

$$P(A \cap A_\varepsilon) \geqslant \frac{\varepsilon}{P(A_0)} \frac{\alpha}{2} \geqslant \frac{\alpha}{2} \varepsilon.$$

Thus $A_\varepsilon$ has the desired properties.

We collect the next four lemmas for the sake of completeness.

LEMMA 4.7. *Let* $\mathbb{N}_1 = \{2^\nu : \nu \in \mathbb{N}\}$ *and* $N_n = \{\nu \in \mathbb{N}_1 : \nu \leqslant [n/\lg n]\}$. *Then*

$$\sum_{\nu \in N_n} \nu^\varepsilon (\lg \nu)^\gamma = \begin{cases} O(n^\varepsilon (\lg n)^{\gamma - \varepsilon}); & \varepsilon > 0,\ \gamma \in \mathbb{R} \\ O((\lg n)^{\gamma + 1}); & \varepsilon = 0,\ \gamma > -1 \\ O(\lg \lg n); & \varepsilon = 0,\ \gamma = -1 \\ O(1); & \varepsilon = 0,\ \gamma < -1. \end{cases}$$

*Proof.* By direct computation.

LEMMA 4.8. *Let* $\mathscr{B}, \mathscr{C} \subset \mathscr{A}$ *be* $\sigma$*-fields.*

(i) *If* $B_n \in \mathscr{B}$, $n \in \mathbb{N}$, *are disjoint then*

$$\sum_{n \in \mathbb{N}} d(B_n, \mathscr{C}) \leqslant 4\rho(\mathscr{B}, \mathscr{C}).$$

(ii) *If* $A \in \mathscr{A}$ *then with* $B = \{P(A \mid \mathscr{B}) > \frac{1}{2}\}$

$$P(A \triangle B) = d(A, \mathscr{B}).$$

*Proof.* Part (i) follows from Theorem 1 of [18]. Part (ii) follows by a direct computation (the idea of using this special set $B$ is due to [20]).

LEMMA 4.9. *Let* $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, *be* i.i.d. *with* $E(X_1) = 0$ *and* *covariance matrix I. Let* $A_\nu := \{|S_\nu| > \rho_3^{1/3}(2k\nu \lg \nu)^{1/2}\}$, *where* $\rho_3 = E(|X_1|^3)$. *Then*

$$\sum_{\nu \in \mathbb{N}_1} \int_{A_\nu} |S_\nu| \, dP \leqslant c(k)\, \rho_3,$$

*where* $\mathbb{N}_1 = \{2^\nu : \nu \in \mathbb{N}\}$.

*Proof.* Follows in the same way as formula (32) in the proof of the $d_1$-inequality of [17] (choose $m(i) = 2^i$).

LEMMA 4.10. *Let* $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, *be* i.i.d. *with* $E(X_1) = 0$ *and covariance matrix I. Then there exists a constant* $c(k)$ *such that for all* $\nu$, $m \in \mathbb{N}$ *with* $\nu \leqslant m/2$,

$$\sup_{C \in \mathscr{C}} \left| P\left(\frac{S_m}{m^{1/2}} \in C \mid X_1, ..., X_\nu\right) - \Phi_{O,I}(C)\right| \leqslant c(k)\, \frac{\rho_3}{m^{1/2}} (\nu^{1/2} + |S_\nu|).$$

*Proof.* Follows directly from Lemma 2 and Remark 3 (ii) of [17].

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